Journal of Pure and Applied Algebra 26 (1982) 221–227
North-Holland Publishing Company

VARIETIES AND ELEMENTARY ABELIAN GROUPS

J.L. ALPERIN*
Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

L. EVENS
Department of Mathematics, Northwestern University, Evanston, IL 60201, USA

Communicated by K.W. Gruenberg
Received 23 Novembver 1981

1. Introduction

Quillen's work [7] on cohomology algebras of finite groups, giving a group-theoretic interpretation of their Krull dimension, was a motivation for our work on module complexity [1], in which we gave a wide generalization to a result on representations of finite groups. Quillen's results on the structure of the associated varieties is the motivation here.

Let G be a finite group and k be a field of prime characteristic p. Let $H(G,k)$ be the subalgebra of the cohomology algebra $H^*(G,k)$ spanned by terms of even degrees if $p\neq 2$ while let $H(G,k)=H^*(G,k)$ if $p=2$. Let X_G be the prime ideal spectrum of $H(G,k)$ endowed, as usual, with the Zariski topology. For any subgroup H of G let ϱ_H be the map of X_H to X_G induced by the restriction map res_H of $H(G,k)$ to $H(H,k)$ (so $\varrho_H(\mathfrak{p})$, for a prime ideal $\mathfrak{p}$ of $H(H,k)$, is the inverse image in $H(G,k)$ under res_H). With this notation, Quillen has in essence shown that

$$X_G=\bigcup_E \varrho_E(X_E)$$

when E runs over all elementary abelian p-subgroups of E.

J.-P. Serre has suggested to us what a proper generalization of this should be and we are pleased to verify his conjectures. We are also indebted to Leonard Scott for suggesting the direction to go and to Judy Sally for much help with the ring-theoretic work that is involved.

For any finitely generated kG-module M let $\mathrm{Supp}_G(M)$ be the set of all $\mathfrak{p}$ in X_G such that the localization $H^*(G,M\otimes S)_{\mathfrak{p}}$ is not zero for some finitely generated kG-module S.

* Supported in part by National Science Foundation Grant MCS-7904469.

0022-4049/82/0000–0000/$02.75 © 1982 North-Holland

Theorem 1. *We have the equality*

$$\operatorname{Supp}_G(M) = \bigcup_E \varrho_E(\operatorname{supp}_E(M_E))$$

where E runs over all elementary abelian p-subgroups of G and M_E is the restriction of M to E.

The result is a consequence of a ring-theoretic result. Let $r_G(M)$ be the ideal in $H(G,k)$ consisting of all elements x such that for all finitely generated kG-modules S there is a positive integer j with $x^j H^*(G, M\otimes S) = 0$.

Theorem 2. *We have the equality*

$$r_G(M) = \bigcap_E \operatorname{res}_E^{-1}(r_E(M_E))$$

as E runs over all elementary abelian p-subgroups E of G.

A similar result has also been established by G. Avrunin [2].

2. Preliminary results

Our notation is as above and before [1]; in particular, all modules are assumed to be finitely generated.

Lemma 2.1. *If H is a subgroup of G and M is a kG-module then $r_G(M) \subseteq \operatorname{res}_H^{-1}(r_H(M_H))$.*

Proof. By the lemma of Eckmann and Shapiro, for each kH-module T we have the isomorphism of $H(G,k)$-modules (see [3])

$$H^*(H, M_H \otimes T) \cong H^*(G, \operatorname{Hom}_{kH}(kG, M_H \otimes T)).$$

Hence, it suffices to show that if $x \in r_G(M)$ then a power of x annihilates the right-hand side. However, the module $\operatorname{Hom}_{kH}(kG, M_H\otimes T)$, which is the induced module, is the tensor product of M and the module induced by T and so our claim follows from the definition of $r_G(M)$.

Lemma 2.2. *If P is a Sylow p-subgroup of G then $r_G(M) = \operatorname{res}_P^{-1}(r_P(M_P))$.*

Proof. In view of the previous result, it suffices to show that $r_G(M) \supseteq \operatorname{res}_P^{-1}(r_P(M_P))$. Hence, let $x \in H(G,k)$ and assume that $x \in \operatorname{res}_P^{-1}(r_P(M_P))$. If S is a kG-module then $M\otimes S$ is a direct summand of the module induced from $M_P \otimes S_P$. Hence, if $\operatorname{res}_P x^i$, $i > 0$, annihilates $H^*(P, M_P \otimes S_P)$ then it certainly annihilates the summand $H^*(G, M\otimes S)$. Thus, $x \in r_G(M)$.

Lemma 2.3. *If*

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence of kG-modules, $x_1, x_2 \in H^*(G,k)$ *annihilates* $H^*(G,M_1)$ *and* $H^*(G,M_2)$, *respectively, then* $x_1 x_2$ *annihilates* $H^*(G,M)$.

Proof. This follows immediately from the long exact sequence connecting the cohomology of M, M_1 and M_2.

A key result of O. Kroll [5] follows from this. Even though we do not require this result in what follows, we pause to make the observation.

Lemma 2.4 (Kroll). *If H is a normal subgroup of prime index p in G, M is a kG-module, A is the annihilator of* $H^*(G,M)$ *in* $H^*(G,k)$ *while B is the annihilator of* $H^*(H,M_H)$ *in* $H^*(H,k)$ *then* $\operatorname{res}_H(A^p) \subset B$.

Proof. Let N be the kG-module induced by M_H so $N \simeq k[G/H] \otimes M$. The cohomology $H^*(H,M_H)$ is a $H^*(G,k)$-module, via restriction, and as such is isomorphic with the $H^*(G,k)$-module $H^*(G,N)$, by the lemma of Eckmann and Shapiro. However, as $k[G/H]$ has a series of submodules with successive quotients isomorphic with the trivial kG-module k, it follows that N has a series of submodules with all its p successive factors isomorphic with M. Thus, the previous result implies that if $x_1, \ldots, x_p \in A$ then $x_1 \cdots x_p$ annihilates $H^*(G,N)$ so we are done.

Lemma 2.5. *If G is a p-group and is not elementary abelian while M is a kG-module then*

$$r_G(M) = \bigcap_H \operatorname{res}_H^{-1}(r_H(M_H))$$

where H runs over all the maximal subgroups of G.

Proof. First, fix a maximal subgroup H of G and let β_H be the inflation to $H^2(G,k)$ of a generator of $H^2(G/H,k)$. We claim that there is a positive integer j, depending only on H, such that if $x \in H(G,k)$ and $\operatorname{res}_H(x) \in r_H(M_H)$ then $x^{2j} H^*(G,M) \subseteq \beta_H H^*(G,M)$.

Let us see that this assertion implies the lemma. By Serre's theorem [9], there exist a sequence $H_1, H_2, \ldots, H_t$ of maximal subgroups of G such that $\prod \beta_{H_i} = 0$. Hence, there is a positive integer N such that if $x \in H(G,k)$ and $\operatorname{res}_{H_i}(x) \in r_{H_i}(M_{H_i})$, for all i, then $x^N H^*(G,M) = 0$. Then, if S is any kG-module, $M \otimes S$ has a filtration by submodules with the successive quotients each isomorphic with M. Hence, by Lemma 2.3, a power of x also annihilates $H^*(G, M \otimes S)$, as required.

In order to establish our assertion, we consider the spectral sequence

$$H^*(G/H, H^*(H,M)) \Rightarrow H^*(G,M)$$

as a module over the spectral sequence

$$H^*(G/H, H^*(H,k)) \Rightarrow H^*(G,k),$$

as in [1]. In particular, each 'column' $H^p(G/H, H^*(H,M))$ is a module over $H^0(G/H, H^*(H,k)) = H^*(H,k)^G$, the invariant subring. Suppose $x \in H(G,k)$ with $y = \mathrm{res}_H x$ and $y \in r_H(M_H)$, so certainly $y \in H^*(H,k)^G$. Hence, there is $j>0$ with $y^j H^*(H,M) = 0$. This implies that $y^j H^p(G/H, H^*(H,M)) = 0$ since G/H is cyclic and $H^p(G/H, H^*(H,M))$ is a quotient of a submodule of $H^*(H,M)$ (the submodule of G/H-invariants if p is even and the submodule annihilated by the norm in $k(G/H)$ if p is odd). Also, since $y^j \in H^0(G/H, H^*(H,k))$, y^j acts on $H^p(G/H, H^*(H,k))$ in the expected way. It now follows that $y^j E_r^{p,*}(M) = 0$ for each $r = 2, 3, \ldots, \infty$. Let

$$H^*(G,M) = F^0 H^*(G,M) \supseteq F^1 H^*(G,M) \supseteq \cdots$$

be the filtration associated with this spectral sequence. Taking $r = \infty$ we get that $x^j(F^p/F^{p+1}) = y^j(E_\infty^{p,*}) = 0$. In particular, $x^{2j} F^0 \subseteq F^2$. But, by Lemma 4.1 of [1] $F^2 H^*(G,M) = \beta_H H^*(G,M)$, so the lemma is proved.

Lemma 2.6. *If* $\mathfrak{p}$ *is a prime ideal of* $H(G,k)$ *then* $\mathfrak{p} \in \mathrm{Supp}_G(M)$ *if, and only if,* $\mathfrak{p}$ *contains* $r_G(M)$.

Proof. First, suppose that $\mathfrak{p} \in \mathrm{Supp}_G(M)$ so there is a kG-module S such that whenever $x \in H(G,k)$, $x \notin \mathfrak{p}$, then $xH^*(G, M \otimes S) \neq 0$. But $x \notin \mathfrak{p}$ implies that $x^i \notin \mathfrak{p}$ whenever $i>0$ so $x \notin r_G(M)$.

On the other hand, suppose $\mathfrak{p} \supseteq r_G(M)$. If $x \notin \mathfrak{p}$ there is a kG-module S such that $x^j H^*(G, M \otimes S) \neq 0$ for all $j>0$. Hence, by Lemma 2.3, there is a simple kG-module S with this property so the same is true if we take S to be the direct sum of simple kG-modules, one of each isomorphism type. This module works for all $x \notin \mathfrak{p}$ so $\mathfrak{p} \in \mathrm{Supp}_G(M)$.

3. Proofs of the main theorems

We begin with the proof of Theorem 2. Since the result is a tautology if G is elementary we assume otherwise. If G is a p-group and H is a maximal subgroup of G then, by induction,

$$r_H(M_H) = \bigcap_E \mathrm{res}_E^{-1}(r_E(M_E)$$

where E runs over all elementary abelian subgroups of H and res_E is the restriction from H to E. Since every elementary abelian subgroup of G is contained in a maximal subgroup of G, we are done in view of Lemma 2.5 and the transitivity property of restriction.

Now let G be arbitrary. If E and E' are conjugate elementary abelian subgroups then $\mathrm{res}_E^{-1}(r_E(M_E)) = \mathrm{res}_{E'}(r_{E'}(M_{E'})$ so we may restrict attention to the elementary abelian subgroups of a fixed Sylow p-subgroup of G. However, Lemma 2.2 and the fact that the theorem holds for p-groups now imply the theorem.

In view of Lemma 2.6, in order to establish Theorem 1, we must show that for each prime ideal $\mathfrak{p}$ of $H(G,k)$ containing $r_G(M)$ there is an elementary abelian p-subgroup E and a prime ideal $\mathfrak{q}$ of $H(E,k)$ containing $r_E(M_E)$ such that $\mathfrak{p} = \mathrm{res}_E^{-1}(\mathfrak{q})$. By Theorem 2, $\mathfrak{p} \supseteq \mathrm{res}_E^{-1}(r_E(M_E))$ for some E. The conclusion now follows from a 'going up' argument as follows.

Let $r_E(M_E) = \bigcap \mathfrak{q}_i$ be the primary decomposition of $r_E(M_E)$ where each $\mathfrak{q}_i$ is a prime ideal since $r_E(M_E)$ is its own radical. Since $\mathfrak{p} \supseteq \mathrm{res}_E^{-1}(r_E(M_E))$ there is i with $\mathfrak{p} \supseteq \mathrm{res}_E^{-1}(\mathfrak{q}_i) = r_i$. Applying the 'going up' theorem to the finite ring extension of $H(G,k)/r_i$ by $H(E,k)/\mathfrak{q}_i$ yields the existence of a prime ideal $\mathfrak{q}$ of $H(E,k)$ containing $\mathfrak{q}_i$, and hence $r_E(M_E)$, with $\mathfrak{p} = \mathrm{res}_E^{-1}(\mathfrak{q})$.

4. Earlier results

In this last section, we shall discuss how Quillen's theorem [7] and our main result on complexity [1] follows from the theorems of this paper. First, Quillen's theorem states that the Krull dimension of $H(G,k)$, denoted $\dim H(G,k)$, equals the maximum of the ranks of the elementary abelian subgroups of G. If E is such a subgroup than it's rank equals $\dim H(E,k)$ as $H(E,k)$ is a finite extension of a polynomial algebra on as many generators as the rank of E. Since $H(E,k)/r_E(k)$ is a finite ring extension of $H(G,k)/\mathrm{res}_E^{-1}(r_E(k))$, it follows that ϱ_E preserves the dimension of closed subspaces. Hence,

$$\begin{aligned}\dim X_G &= \dim \bigcup_E \varrho_E(X_E)\\ &= \max_E \dim \varrho_E(X_E)\\ &= \max_E \dim X_E\end{aligned}$$

as required.

Before proceeding to the complexity result we want to sketch how one may prove Quillen's theorem very directly without invoking any geometric concepts. We rely on the characterization of the Krull dimension of a graded ring of the type we have in terms of the growth rate (Definition 2.1 of [1]) of the homogeneous components. (See the appendix below for a proof.) One first proves that $\dim H(G,k) \geq \dim H(H,k)$ for any subgroup H of G by relying on the fact that the latter is a finite module over the former [3]. Similarly, one proves $\dim H(G,k) = \dim H(P,k)$ for a Sylow p-subgroup. Finally, one shows for a p-group G, which is not elementary abelian, that $\dim H(G,k)$ equals the maximum of the $\dim H(H,k)$, as H runs over the maximal subgroups of G, as follows. For each maximal subgroup H of G note that the growth rate of $F^0H^*(G,k)/F^2H^*(G,k)$ is bounded by that of $\bigoplus_{p=0,1} H^p(G/H,H^*(H,k))$ and hence by the growth rate of $H(H,k)$. The equality $F^2H^*(G,k)=\beta_H H^*(G,k)$ and Serre's result on products of Bocksteins show that the filtration

$$H^*(G,k)\supseteq \beta_{H_1}H^*(G,k)\supseteq \beta_{H_2}\beta_{H_1}H^*(G,k)\supseteq \cdots$$

terminates.

In view of Theorem 1, to prove the theorem on complexity, we need only show that $\dim \mathrm{Supp}_G(M)$ equals the complexity of M. Let S be the direct sum of simple kG-modules, one of each isomorphism type. Applying the appendix below to the $H(G,k)$-module $H^*(G,M\otimes S)$, we see that the Krull dimension of $H(G,k)/r_G(M)$ equals the growth rate of $\dim G^k(G,M\otimes S)$ – which is the complexity.

5. Appendix

The characterization of Krull dimension in terms of growth rates seems to be known to workers in this area but as no proof of the *precise* result we want is available in the literature we shall give ours. [See Smoke [10], Theorem 5.5 and Matijevic [6], Theorem 1.2 for closely related facts.]

Fix a finitely generated commutative algebra A over the field k which is graded over the non-negative integers with $A_0=k$.

Lemma 5.1. *There is a positive integer N and rational polynomials $f_0,f_1,\dots,f_{N-1}$ such that, with finitely many exceptions,*

$$\dim A_n=f_r(n)$$

whenever $n\equiv r$ (modulo N).

Proof. Let $z_1,\dots,z_s$ be a set of generators for A each of which is homogeneous of a positive degree. Let N be the least common multiple of the degrees of the z_i. Let w_i be the power of z_i which has the degree N. Hence, A is a finitely generated module over $K[w_1,\dots,w_s]$. Let

$$A^{(i)}=\bigoplus_{j=0}^{\infty} A_{i+jN}$$

for $i=0,1,\dots,N-1$. Hence $A^{(i)}$ is a finitely generated graded module for $K[w_1,\dots,w_s]$ with A_{i+jN} the homogeneous summand of degree j. The result now follows from the theorem of Hilbert–Serre [11, p. 232 of Volume II] applied to each $A^{(i)}$.

Now let B be the vector space over k, graded over the non-negative integers, with $B_n=\bigoplus_{i=0}^{n} A_i$. Lemma 5.1 yields immediately that (with γ as in [1]) $\gamma(B)=\gamma(A)+1$. Hence, we need only show that $\gamma(B)$ equals the Krull dimension d of A. Since A is of dimension d there is a polynomial subalgebra $k[x_1, \ldots, x_d]$ of A finitely generated as a module over it. Let $y_1, \ldots, y_e$ be module generators.

First, we show that $\gamma(B) \geq d$. Let M be a positive integer such that the i-th component of every x_j is zero if $i > M$. We may also assume, without loss of generality that the zero component of each x_j is also zero, so with the obvious notation

$$x_j = x_{j1} + \cdots + x_{jM}.$$

Hence, all the monomials

$$x_1^{a_1} \cdots x_d^{a_d}$$

with $M(a_1 + \cdots + a_d) \leq n$ are linearly independent and lie in B_n. This proves the desired inequality.

Finally, we must prove that $\gamma(B) \leq d$. It suffices to describe a spanning set for A, take the projection of this set on $A_0 \oplus \cdots \oplus A_n = B_n$ for each n, count the number of non-zero projections and have this number of the right size. For the spanning set we take all monomials

$$x_1^{a_1} \cdots x_d^{a_d} y_j,$$

$1 \leq j \leq e$. In order to have a non-zero projection on B_n we must have $a_1 + \cdots + a_d \leq n$, since the zero components of each x_i is zero; this proves the final inequality.

We have just seen that the growth $\gamma(A)$ of the A-module A is $d-1$. It follows that if M is a faithful finitely generated graded A-module then $\gamma(M) = d-1$. Indeed, M is a homomorphic image of a free module so $\gamma(M) \leq d-1$. Moreover, if M has (homogeneous) generators $m_1, \ldots, m_r$ then the A-module A is isomorphic with a submodule of $M \oplus \cdots \oplus M$ (r copies of M with $a \in A$ mapped to $(am_1, \ldots, am_r)$) so $\gamma(M) \geq d-1$.

References

[1] J. Alperin and L. Evens, Representations, resolutions and Quillen's dimension theorem, J. Pure Appl. Algebra 22 (1981) 1–9.

[2] G. Avrunin, Annihilators of cohomology modules, J. Pure Appl. Algebra, to appear.

[3] L. Evens, The cohomology ring of a finite group, Trans. Amer. Math. Soc. 101 (1961) 224–239.

[4] J. Johnson and J. Matijevic, Krull dimension in grade rings, Comm. Algebra 5 (1977) 319–329.

[5] O. Kroll, Complexity and Elementary Abelian Subgroups, Thesis, University of Chicago (1980).

[6] J. Matijevic, Some Topics in Graded Rings, Thesis, University of Chicago (1973).

[7] D. Quillen, A cohomological criterion for p-nilpotence, J. Pure Appl. Algebra 1 (1971) 361–372.

[8] D. Quillen and B. Venkov, Cohomology of finite groups and elementary abelian subgroups, Topology 11(1972) 317–318.

[9] J.P. Serre, Sur la dimension cohomologique des groupes profinis, Topology 3 (1965) 413–420.

[10] W. Smoke, Dimension and multiplicity for graded algebras, J. Algebra 21 (1972) 149–173.

[11] O. Zariski and P. Samuel, Commutative Algebra, Vol. II (Van Nostrand, New York, 1960).